

# Stability Analysis of the Ant System Dynamics with Non-uniform Pheromone Deposition Rules

Ajith Abraham, Amit Konar, Nayan R. Samal and Swagatam Das

**Abstract**—The paper extends the classical Ant Systems by considering non-uniform deposition by the ants, while constructing pheromone trails. A deterministic solution to the ant system dynamics for both uniform and non-uniform pheromone deposition rules has been obtained to determine the parameters of the dynamics that ensure stability in pheromone trails. Computer simulation confirmed the results of stability analysis. Performance of the extended ant system (with non-uniform pheromone deposition rule) is compared with the classical ant system using the well known Traveling Salesperson Problem. Simulation results reveal that the extended ant system outperforms the classical ant system by a large margin with respect to convergence speed without sacrificing the quality of solution.

## I. INTRODUCTION

French entomologist Pierre-Paul Grassé [1] in the mid-1950's first observed that some species of termites react to external stimulus produced by other members of the colony. This particular type of communication among the members of the colony is referred to as *stigmergy* in the literature [2]. Two fundamental characteristics of stigmergy that differentiate it from the other forms of communication include: i) exchange of non-symbolic information among insects by modifying their environment, and ii) local changes in environment influence insects in the neighborhood [3]. The principles of communication by stigmergy are prevalent in many species of ants. While walking to and from a food source, ants release pheromones on the ground. The members of the ant colony follow the paths of higher pheromone concentration. Denebourg *et al.* [4] studied the pheromone laying and following behavior of ants. Their work is the main source of inspiration for the ant system [5]-[7] and ant colony optimization (ACO) [8]-[9] in particular.

Among the well known ACO algorithms, the *Ant System* (AS) [5], *Max-Min Ant System* (MMAS) [10] and the *Ant Colony System* (ACS) [11], [12] and [13] are most popular. Ant System is the first ACO algorithm proposed

in the literature [5]-[7]. MMAS and ACS are variants of the classical AS. In the classical AS, the pheromone values are updated by all the ants that have built a solution in the current iteration. The classical AS employs a uniform pheromone rule. For example, suppose the trial solution generated by an ant for a given problem corresponds to a singly-connected graph of length  $L$ . Assuming that the ants synthesize a maximum volume of pheromone  $Q$ ; the deposition rate over the trial solution graph of length  $L$  is  $Q/L$ . This may be regarded as uniform deposition of pheromone over the trail solution track. In this paper, we extend AS by considering  $Q$  being a time varying non-decreasing function  $Q(t)$ , and considering different mathematical forms of  $Q(t)$ . We would like to examine whether such non-decreasing  $Q(t)$  improves the expected convergence time of the algorithm. The apparent correlation between the selection of non-decreasing  $Q(t)$  and the expected convergence time of extended AS is indicated below.

A uniform pheromone deposition by an ant (while constructing trial solution in program iteration) cannot ensure subsequent ants to follow the same trajectory. A non-uniform non-decreasing time function  $Q(t)$ , i.e. the first derivative over time is  $\geq 0$ , however, ensures that subsequent ants close enough to a previously selected trial solution will follow the trajectory, as it can examine gradually thicker deposition of pheromones over the trajectory. Naturally, *deception probability* [14] of the ants will be less, consequently improving expected convergence time.

To study the effect of non-uniform pheromone deposition rule, we consider the ant system dynamics and solved it for different  $Q(t)$  including the case for classical

ant systems where  $Q(t) = \bar{C}$ , constant. We then examine the stability of the AS dynamics from the deterministic solution of the dynamics. This gives us the range of parameters for the dynamics, confirming stability of the AS. It is intuitively apparent that an unstable ant system dynamics may result in an exponentially uncontrolled growth in pheromone deposition, causing over-exploitation of the ants. Such exploitation has no apparent mathematical algorithmic benefits. A stable ant dynamics, however, confirms convergence of the ant system and prevents undesirable exploitation of the ants.

Most of the exciting works on ACO are simulation-based. There exists limited research results on the

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theoretical/foundations on ACO. The first convergence proof to ACO was undertaken by Gutjahr [15] for a graph based ant system (GBAS). Dorigo and Stützle [16], [17] provide a convergence proof for the ACS and MMAS algorithms. Theoretical predictions about the speed of convergence of specific ACO algorithms are undertaken in [18]. Blum and Dorigo [19], [20] demonstrated that ACO algorithms generally suffer from *first order deception* problem. The first study on the behavior of ACO algorithms by analyzing the dynamics of the pheromone model was undertaken in [14]. They have shown that in idealized permutation problems, constraints on the feasibility of solutions introduce *selection bias* in the solution construction process. The differential equation approach to solve the deterministic ant system dynamics undertaken in this paper is novel. This helps in determining the range of parameters in the non-uniform pheromone deposition rule to confirm stability in pheromone trails.

The deterministic solution to the ant system dynamics undertaken here does not violate the stochastic nature of the AS. This is due to the fact that a segment of path (trajectory) here too is selected probabilistically.

Rest of the paper is organized as follows. In Section II, we formulate a scheme for the general solution of the AS. Closed form solutions for stability analysis for different pheromone deposition rule are undertaken in Section III. Performance analyses of the proposed and classical AS are compared in Section IV. Conclusions are listed in Section V.

## II.A DETERMINISTIC FRAMEWORK FOR THE SOLUTION OF THE BASIC ANT SYSTEM DYNAMICS

To start with, we consider a small segment of the tour by an ant. Let  $i$  and  $j$  be two successive nodes, on the tour of an ant, and  $\tau(i, j, t)$  is the pheromone concentration associated with the edge of the graph (trail solution) created by the ant at time  $t$  (Figure 1).

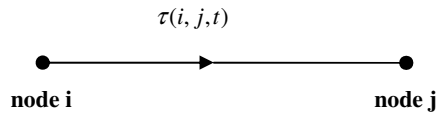


Fig. 1. Defining  $\tau(i, j, t)$

Let  $\rho$  be the pheromone evaporation rate, and  $\Delta\tau_k(i, j, t)$  be the pheromone deposited by ant  $k$  at time  $t$ . The basic pheromone updating rule is then given by

$$\tau(i, j, t) = (1 - \rho)\tau(i, j, t-1) + \sum_{k=1}^m \Delta\tau_k(i, j, t) \quad (1)$$

where  $\rho > 0$ .

The solution of the ant dynamics is outlined in this paper by solving the difference equation (1). We segregate the complimentary function and the particular integral. In fact, we consider different forms of  $\Delta\tau_k(i, j, t)$  and attempt to determine the closed form solution to  $\tau(i, j, t)$ .

### A. Evaluating the Complimentary Function (CF)

The complimentary function of equation (1) is obtained by setting the right hand side of equation (1) to zero. This gives only the transient behavior of the ant dynamics.

$$\tau(i, j, t) - (1 - \rho)\tau(i, j, t-1) = 0 \quad (2)$$

Let  $E$  be the extended difference operator [21], where for any real  $x$ ,  $x + \Delta x = (1 + \Delta)x = E(x)$  and thus  $E \equiv 1 + \Delta$ . Consequently, we formulate:

$$\tau(i, j, t-1) = E^{-1}\tau(i, j, t). \quad (3)$$

Now, from equations (2) and (3), we obtain:

$$\tau(i, j, t) - (1 - \rho)E^{-1}\tau(i, j, t) = 0$$

which implies that  $\{E - (1 - \rho)\}/E = 0$

$$\text{or, } E = (1 - \rho) \quad (4)$$

Thus, the transient behavior of the Ant System is given by

$$\text{CF: } \tau(i, j, t) = A(1 - \rho)^t \quad (5)$$

where  $A$  is a constant, which is to be obtained later from initial conditions. We now consider different variations of  $\Delta\tau_k(i, j, t)$  and obtain the particular integral in the following sub-section.

### B. Evaluating the Particular Integrals for Different Forms of Pheromone Trail Construction

The steady state solution of the ant dynamics may be obtained by computing particular integral. In this study, we consider different forms of pheromone deposition. We divide them into three cases depending on their type of variations in pheromone constructs.

Case I:  $\Delta\tau_k(i, j, t) = \bar{c}$ , constant, where  $\bar{c} > 0$ ,

Case II:  $\Delta\tau_k(i, j, t) = a_k^t$ , where  $a_k > 0$  for all  $k$ ,

Case III:  $\Delta\tau_k(i, j, t) = k(1 - e^{-t/\tau})$  for  $k > 0$  and  $\tau > 0$ ,

The particular integral (PI) for the Ant System is obtained by re-organizing (1) as follows:

$$[1 - (1 - \rho)E^{-1}]\tau(i, j, t) = \sum_{k=1}^m \Delta\tau_k(i, j, t) \quad (6)$$

$$\text{or, } \tau(i, j, t) = \frac{1}{1 - (1 - \rho)E^{-1}} \sum_{k=1}^m \Delta\tau_k(i, j, t) \quad (7)$$

### III. CLOSED FORM SOLUTION AND STABILITY ANALYSIS

In this Section, we obtain the closed form solution of the ant system dynamics (equation (1)) for determining the condition for stability of the dynamics. All the 3 cases outlined in Section II are studied further to determine the range of parameters in each case, confirming the stability of the dynamics.

**Case I:** When  $\Delta\tau_k = \bar{c}$ , i.e. uniform, we obtained from equation (7)

$$\tau(i, j, t) = \frac{1}{1 - (1 - \rho)E^{-1}} \sum_{k=1}^m \bar{c} \quad (8)$$

Since  $\bar{c} = \text{constant}$ , we set  $E = 1$  in equation (8), and thus obtain

$$\begin{aligned} \tau(i, j, t) &= \frac{1}{1 - (1 - \rho)} m \bar{c} \\ &= \frac{m \bar{c}}{\rho} \end{aligned} \quad (9)$$

This is the steady-state value of  $\tau(i, j, t)$ . Combining the PI from equation (9) with the CF obtained in equation (5), we finally have,

$$\tau(i, j, t) = A(1 - \rho)^t + \frac{m \bar{c}}{\rho}. \quad (10)$$

When  $t=0$ , we find from equation (10)

$$A = \tau(i, j, 0) - \frac{m \bar{c}}{\rho}. \quad (11)$$

Substituting the value of A from (11) to expression (10), we find

$$\tau(i, j, t) = \left[ \tau(i, j, 0) - \frac{m \bar{c}}{\rho} \right] (1 - \rho)^t + \frac{m \bar{c}}{\rho}. \quad (12)$$

It is apparent from equation (12) that as  $\rho < 1$ , the first term in the R.H.S. of (12) reduces to zero, and we obtain a steady-state value  $\frac{m \bar{c}}{\rho}$ .

What does the result indicate? It means that the steady-state value of  $\tau(i, j, t)$  is proportional to  $\bar{c}$  and  $m$  and reciprocal to  $\rho$ . Figure 2 provides plot of  $\tau(i, j, t)$  versus  $t$  for varying  $\rho$ , which too supports the above observation. So, the condition for stability is  $\rho < 1$ .

**Case II:** With  $\Delta\tau_k = a_k^t$ , i.e. parabolic we obtain

$$PI = \frac{1}{1 - (1 - \rho)E^{-1}} \sum_{k=1}^m a_k^t \quad (13)$$

As  $a_k = \text{constant}$ , we set  $E = a_k$  and thus obtain

$$PI = \sum_{k=1}^m \frac{a_k^{t+1}}{a_k - 1 + \rho} \quad (14)$$

Let  $a_k = \bar{a} - \rho$  for all  $k = 1$  to  $m$ .

Then

$$PI = m \frac{(\bar{a} - \rho)^{t+1}}{\bar{a} - 1} \quad (15)$$

The complete solution to the ant system with parabolic deposition rule is obtained like case I, and is given by

$$\tau(i, j, t) = \left[ \tau(i, j, 0) - m \frac{(\bar{a} - \rho)}{\bar{a} - 1} \right] (1 - \rho)^t + m \frac{(\bar{a} - \rho)^{t+1}}{\bar{a} - 1} \quad (16)$$

The following observations readily follow from equation (16).

- (i) when  $\rho < 1$  and  $0 < |\bar{a} - \rho| < 1$ , the dynamics is stable with a steady state value zero.
- (ii) when  $\rho < 1$  and  $|\bar{a} - \rho| = 1$ , the dynamics is still stable with non-zero steady-state value  $= m/(\bar{a} - 1)$ .
- (iii) when  $\rho < 1$  and  $\bar{a} - \rho = -1$ , the dynamics is oscillatory.
- (iv) when  $|\bar{a} - \rho| > 1$ , the dynamics is unstable.

Figures 3(a), 3(b) and 3(c) provide plots of  $\tau(i, j, t)$  versus  $t$  for varying  $\rho$ , which too support the above observations. So, the condition for stability is  $\rho < 1$  and  $|\bar{a} - \rho| \leq 1$ .

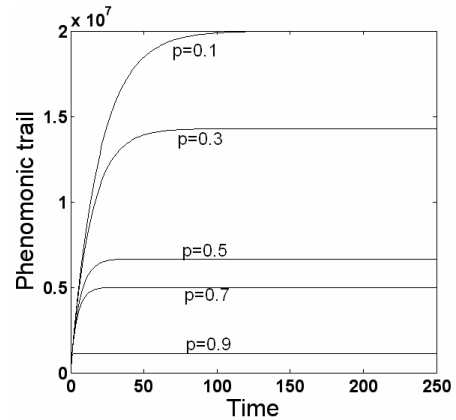


Fig.2.  $\tau(i, j, t)$  versus  $t$ . Stable behavior of the dynamics for constant pheromone deposition. With increase in  $\rho$ , steady state value of  $\tau$  diminishes

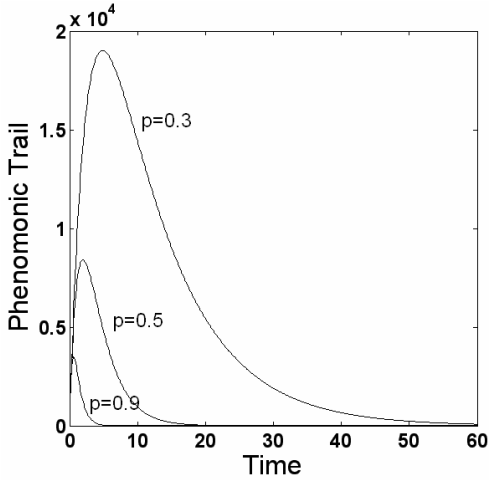


Fig.3. (a).  $\tau(i, j, t)$  versus  $t$  for  $\rho < 1$  and  $|\bar{a} - \rho| < 1$ . Dynamics is stable with a steady state value zero

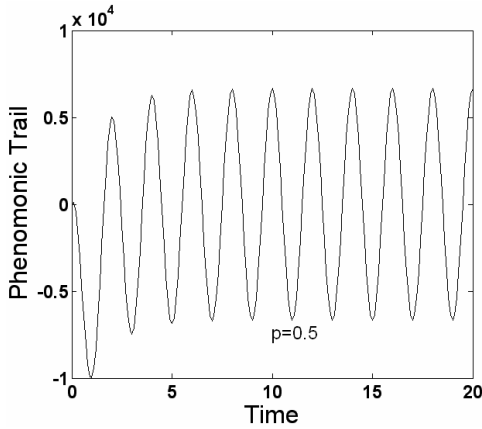


Fig.3.(b).  $\tau(i, j, t)$  versus  $t$  for  $\rho < 1$  and  $\bar{a} - \rho = -1$ . The dynamics is oscillatory

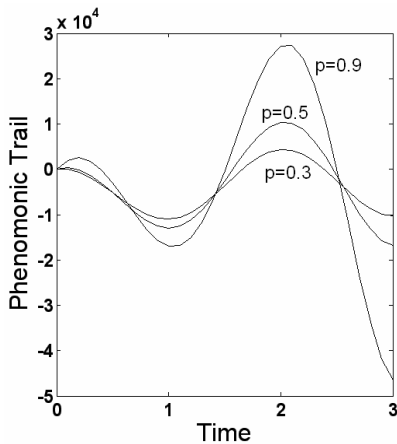


Fig.3.(c).  $\tau(i, j, t)$  versus  $t$  for  $|\bar{a} - \rho| > 1$  The dynamics is unstable

**Case III:** When  $\Delta\tau_k = \bar{c} (1 - e^{-\frac{t}{\tau}})$ , we obtain from equation (7)

$$PI = \frac{1}{1 - (1 - \rho)E^{-1}} \sum_{k=1}^m \bar{c} (1 - e^{-\frac{t}{\tau}})$$

$$= \frac{\bar{m}\bar{c}}{\rho} - \frac{\bar{m}\bar{c}.e^{-(1-t)/\tau}}{e^{1/\tau} + (\rho - 1)} \quad (17)$$

The complete solution to AS can be obtained by summing the CF and PI, and is given by

$$\tau(i, j, t) =$$

$$\left[ \tau(i, j, 0) - \frac{\bar{m}\bar{c}}{\rho} + \frac{\bar{m}\bar{c}.e^{1/\tau}}{e^{1/\tau} + (\rho - 1)} \right] (1 - \rho)^t$$

$$+ \frac{\bar{m}\bar{c}}{\rho} - \frac{\bar{m}\bar{c}.e^{-(1-t)/\tau}}{e^{1/\tau} + (\rho - 1)} \quad (18)$$

It is apparent from equation (18) that the dynamics is stable for  $0 < \rho < 1$ , and the steady-state value of  $\tau(i, j, t)$  is  $\frac{\bar{m}\bar{c}}{\rho}$ .

The smaller the values of  $\rho$  and/or  $\tau$ , the larger the steady-state value of  $\tau(i, j, t)$ . Figure 4 illustrates  $\tau(i, j, t)$  versus  $t$  for varying  $\rho$  and  $\tau$ , which too supports the above observation.

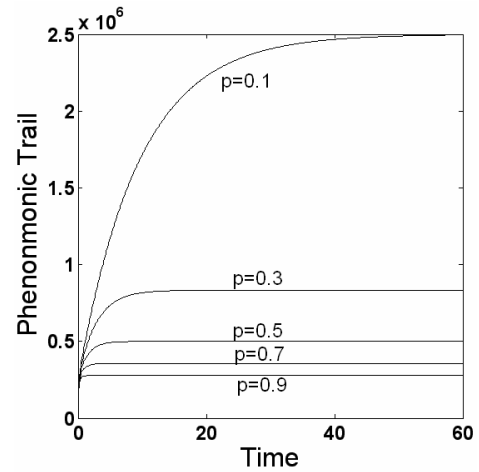


Fig.4. Steady state analysis for exponential function, for  $\tau = 0.1$

IV. PERFORMANCE ANALYSIS OF THE EXTENDED ANT SYSTEM USING TRAVELING SALESPERSON PROBLEM

A computer simulation of the well known Traveling Salesperson Problem (TSP) is undertaken to compare the performance of the extended AS with the classical one. Here, we considered 30 and 35 cities randomly distributed on a given 2-dimensional space. The algorithm is initialized with  $n$  ants, each one of which is located in one of the  $n$ -cities, and no city is devoid of ants. In a given program iteration, an ant from its home city starts its journey, traverses all the cities following the AS based ACO algorithm and finally returns home city. After it returns home, the pheromone on the path traversed by the ant is updated following equation (1). To compare the merits of the extended AS, we here select different mathematical forms of  $\Delta\tau_k$ , as mentioned under the 3 cases in section 3, the first one being the classical one. If  $L$  is the length of the tour by an ant, we set  $\bar{c} = Q/L$  in case I, where  $Q$  is the pheromone synthesized by ant before traversal. In the same manner, we set  $\Delta\tau_k = a_k^i / L$  in case II and  $Q/L(1 - e^{-\frac{t}{\tau}})$  in case III respectively.

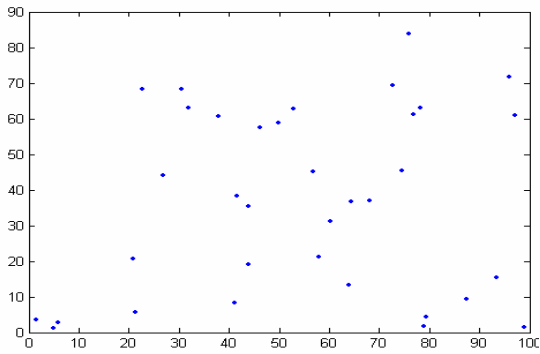


Fig. 5 (a). City Distribution 1

After all ants in a given iteration returns home, we determine the best path (i.e. the path with the shortest length) in the iteration. This is called best tour length in this paper. The iterations are repeated until two successive iterations yield the same length for the best path.

Two sample city distributions and respective plots for best tour length versus iterations are given in Figures 5 and 6. The plots reveal that the exponential AS outperforms the parabolic AS and both the extended AS algorithms outperform classical AS with reference to convergence time.

1000 independent runs were taken for both classical and extended AS algorithms with a random distribution of cities, and the results are more or less uniform. This proves the robustness of the extended AS.

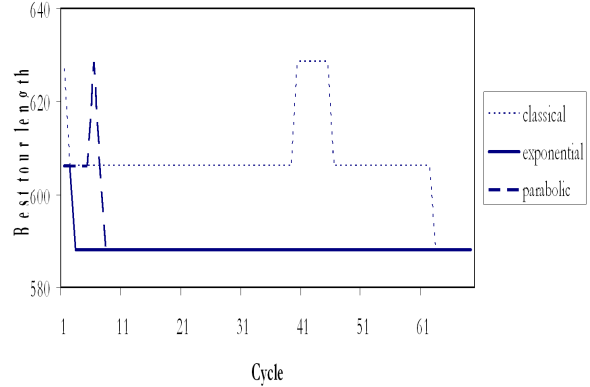


Fig. 5 (b). A sample plot of best tour length for the classical AS and its two extended (parabolic and exponential) versions over iterations (cycles) for random city distribution 1

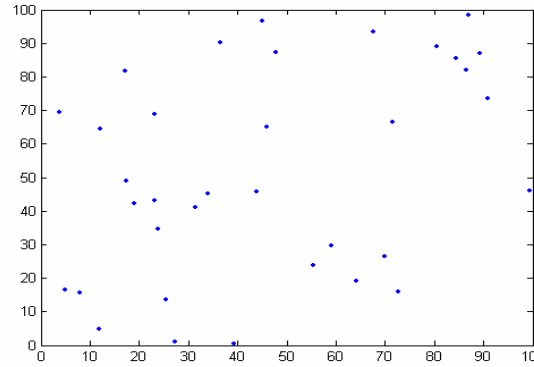


Fig. 6 (a). City distribution 2

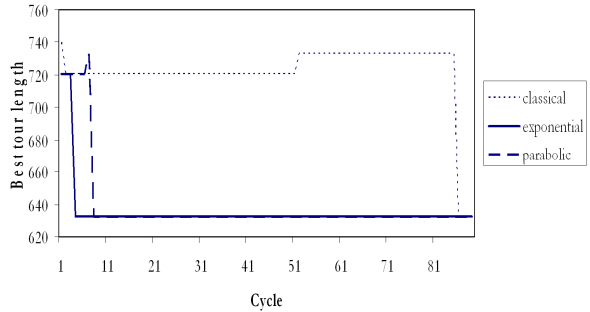


Fig. 6 (b). A sample plot of best tour length for the classical AS and its two extended (parabolic and exponential) versions over iterations (cycles) for random city distribution 2

The parameter set for the best performance of the extended AS algorithms that minimizes the convergence time is  $\tau = 10$  and  $\bar{C} = K = 100$  for exponential deposition rule, and  $a_k = 1.6$  and  $\rho = 0.7$  for parabolic deposition rule. All

computer simulations are performed for the reported best parameter set of the classical AS algorithm [16].

## V. CONCLUSIONS

The paper proposed a simple extension of the Classical AS by considering two different non-uniform pheromone deposition rules. Stability analysis of the AS for both uniform (case I) and non-uniform deposition rules (case II and case III) have been undertaken, and the analysis reveals the range of parameters for convergence of the ant dynamics to fixed stable points.

Computer simulation on TSP demonstrates that the extended AS with non-uniform pheromone deposition rules converge much faster than the classical AS with uniform pheromone deposition rule. It is further noted that the AS employing exponential pheromone deposition rule yields better results in comparison to the one employing parabolic deposition rule. The parameters for the exponential pheromone deposition rule giving the best results (i.e. earliest convergence without losing accuracy in solution) include  $\tau = 10$  and  $K = 100$ .

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